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Singular Points and Hermite Series

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1. Eigenfunction expansions serve as a useful tool in determining properties of the functions to which they converge. In the case of analytic functions an important property is the location of singular points. Nehari [1] attacked the problem of determining the location of a singular point of an analytic function given by a Legendre series. His result was extended to functions given by Sturm-Liouville series by Howard and Gilbert [2] and to nonanalytic functions and distributions by the author [3].

In this paper we consider entire functions and their expansion in terms of Hermite series. The property we shall study is the behavior at infinity of the functions in place of the location of a singular point.

The approach used in the references above is to compare the location of a singular point of the given series to a singular point of an associated power series. The same procedure will be used here except that the behavior at infinity will be compared to the location of the singular point of the series. More exactly, it will be shown (1) that for series of Hermite polynomials $\sum a_n H_n/n!$ which converge to entire functions the behavior at infinity depends on the location of singular points of $\sum a_n t^n$, (2) that for series of Hermite functions $\sum c_n h_{2n}(0)\, h_{2n}(x^{1/2})$ which converge to entire functions the behavior at infinity is reflected in the location of singular points of $\sum c_n t^n$.

2. In this section we consider *series of Hermite polynomials* which converge to entire functions. We use the well-known generating function

$$e^{2xt-t^2} = \sum \frac{H_n(x)\, t^n}{n!} \tag{1}$$

as a basic tool. Throughout this section $\{a_n\}$ is a sequence of complex numbers such that $0 < \overline{\lim}\, | a_n |^{1/n} < \infty$. Then we shall see that

$$f(x) = \sum \frac{a_n}{n!} H_n(x)$$

is an entire function of exponential type. Its behavior at infinity is reflected in its indicator diagram which in turn depends on the location of the singular points of its Borel transforms (see B. Ya. Levin [4] p. 85). Our first result tells where these singular points are.

THEOREM 1. *The Borel transform* $\phi(z)$ *of*

$$f(x) = \sum \frac{a_n H_n(x)}{n!}$$

has a singular point at $z = \alpha$ *if and only if* $\psi(t) = \sum a_n t^n$ *has one at* $t = 2/\alpha$.

We first find an integral representation of ψ in terms of ϕ and vice versa and then employ the "multiplication of singularities" trick to reach the conclusion. Indeed $f(x)$ is related to ψ by

$$\begin{aligned} f(x) &= \sum a_n \frac{H_n(x)}{n!} = \sum_n \frac{H_n(x)}{n!} \sum_k a_k \frac{1}{2\pi i} \int_\Gamma t^{k-n} \frac{dt}{t} \\ &= \frac{1}{2\pi i} \int_\Gamma \sum_n \frac{H_n(x)\, t^{-n}}{n!} \sum_k a_k t^k \frac{dt}{t} \\ &= \frac{1}{2\pi i} \int_\Gamma \exp(2xt^{-1} - t^{-2})\, \psi(t) \frac{dt}{t}, \end{aligned} \tag{2}$$

where Γ is any circular path with center at the origin with radius less than the radius of convergence of $\sum a_n t^n$. Under these conditions the convergence of both series is uniform in t whence the implicit interchange of summation and integral in (2) is allowed. (Note : the asymptotic formula for $H_n(x)\, e^{-x^2/2}$ is $\sqrt{n!}\, 2^{n+1/2} n^{-1/2} \pi^{-1/2} \cos((2n+1)\, x^{1/2} - n\pi/2)$ as $n \to \infty$). (see [5], p. 198).

This formula (2) shows f is of exponential type. Its Borel transform (see [4], p. 85) then is given by

$$\begin{aligned} \phi(z) = \int_0^\infty e^{-zx} f(x)\, dx &= \frac{1}{2\pi i} \int_0^\infty e^{-zx} \int_\Gamma \exp(2xt^{-1} - t^{-2}) \frac{\psi(t)\, dt}{t}\, dx \\ &= \frac{1}{2\pi i} \int_\Gamma \int_0^\infty e^{x(2/t - z)}\, dx e^{-1/t^2} \psi(t) \frac{dt}{t} \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{1}{zt - 2} e^{-1/t^2} \psi(t)\, dt, \end{aligned} \tag{3}$$

where the operations made are valid if z is any point such that $2\mathcal{Re}\, 1/t < \mathcal{Re}\, z$ for each t on the contour Γ. The function $\phi(z)$ can then be extended analytically from such z to all z which satisfy $2/|t| < |z|$ for all t on the contour by using the last expression, since, in that case,the singularity from $1/(zt - 2)$ lies inside the contour. Moreover, $\phi(z)$ can even be extended to points such that $zt - 2 = 0$ for t on the contour by deforming the contour outward to avoid the point $2/z$. This works wherever the function

$\psi(t)$ is holomorphic. However if $\psi(t)$ has a singularity at a certain point the contour becomes "pinched" between this point and the moving singularity from $1/(zt-2)$. Thus the only candidate for a singular point of $\phi(z)$ is a common singular point of $\psi(t)$ and $1/(zt-2)$. If ψ has a singularity at $2/\alpha$ only, the only possible singularity of ϕ is at α. This is the Hadamard "multiplication of singularities" argument.

We now try the same procedure in the other direction, that is find an operator giving ψ in terms of ϕ. We note that the Borel transform of the entire function

$$g(t) = \sum \frac{a_n t^n}{n!}$$

is given by the series $\sum a_n t^{-n-1} = t^{-1}\psi(t^{-1})$. But we also may express g as

$$\begin{aligned} g(t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sum_n \frac{H_n(x)}{n!} \frac{t^n}{2} \sum_k \frac{a_k H_k(x)}{k!} e^{-x^2}\, dx \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{xt-t^2/4} e^{-x^2} f(x)\, dx \end{aligned} \tag{4}$$

as a consequence of the orthogonality of the H_n with respect to e^{-x^2}. (Note: $\int_{-\infty}^{\infty} H_n^2(x)\, e^{-x^2}\, dx = 2^n n! \pi^{1/2}$). Thus we may calculate $t^{-1}\psi(t^{-1})$, by using the integral expression for the Borel transform, to be

$$\int_0^{\infty} e^{-t\xi} g(\xi)\, d_\xi = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \int_{-\infty}^{\infty} e^{x\xi - \xi^2/4 - x^2 - t\xi} f(x)\, dx\, d\xi. \tag{5}$$

Since f is related to its Borel transform ϕ by

$$f(x) = \frac{1}{2\pi i} \int_C e^{xz} \phi(z)\, dz,$$

where C is any contour enclosing the conjugate diagram of f (see [4], p. 85), we see that

$$\begin{aligned} t^{-1}\psi(t^{-1}) &= \frac{1}{2\pi i \sqrt{\pi}} \int_0^{\infty} \int_{-\infty}^{\infty} \int_C e^{x\xi - \xi^2/4 - x^2 - t\xi + xz} \phi(z)\, dz\, dx\, d\xi \\ &= \frac{1}{2\pi i} \int_C \int_0^{\infty} e^{-t\xi - \xi^2/4} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-ix(i\xi + iz) - x^2}\, dx \phi(z)\, d\xi\, dz \\ &= \frac{1}{2\pi i} \int_C \int_0^{\infty} e^{-t\xi - \xi^2/4} e^{(\xi + z)^2/4} \phi(z)\, d\xi\, dz \\ &= \frac{1}{2\pi i} \int_C \frac{1}{t - \frac{z}{2}} e^{z^2/4} \phi(z)\, dz. \end{aligned} \tag{6}$$

Here the uniform convergence sufficient for the indicated manipulation is immediate if $\mathcal{R}e\, t > \mathcal{R}e\,(z/2)$ since the integrand is dominated by a constant $\cdot\, e^{-[(\xi/2)-x]^2-\xi t+xz}$. The same argument as above enables us to conclude that $\psi(t^{-1})$ has a possible singularity only at $t^{-1} = 2/\alpha$ if ϕ has one at α. Thus the Theorem is proved.

3. We now turn to *series of Hermite functions* in place of Hermite polynomials. In order to investigate them we introduce another generating function. It is

$$\sum_{n=0}^{\infty} h_{2n}(0)\, h_{2n}(x)\, t^n = \frac{1}{\sqrt{2\pi}}\, e^{(x^2/2)(t+1)/(t-1)},$$

$$|\, t\,| < 1, \quad x \in (-\infty, \infty), \tag{7}$$

where $h_n(x)$ is the normalized Hermite function

$$h_n(x) = \frac{H_n(x)\, e^{-(x^2/2)}}{A_n}$$

which satisfies $\int_{-\infty}^{\infty} h_n{}^2 = 1$.

It is proved by taking the Fourier transform with respect to x of both sides of

$$\sum_{n=0}^{\infty} \frac{x^{2n} e^{-(x^2/4)} t^n}{4^n n!} = e^{x^2(t-1)/4}; \qquad |\, t\,| < 1$$

to get, by using the fact that $e^{-x^2} H_n(x) = (-1)^n D^n e^{-x^2}$, that

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}(x)\, e^{-x^2} t^n}{4^n n!} = e^{x^2/(t-1)}.$$

But the normalizing factor for H_n is such that

$$H_{2n}(x) = (-4)^n\, n!\, h_{2n}(0)\, h_{2n}(x)\, e^{x^2/2} \sqrt{2\pi},$$

whence we get (7).

Our result about normalized Hermite series, again deals with the growth of the function f at ∞.

THEOREM 2. *Let* $\{a_n\}$ *be a sequence such that*

$$\overline{\lim}\, |\, a_n\,|^{1/n} < 1;$$

then $f(x) = \sum a_n h_{2n}(0)\, h_{2n}(x^{1/2})$ *is an entire function of exponential type whose Borel transform* $\phi(z)$ *has a singular point at* $z = \frac{1}{2}(1+\alpha)/(1-\alpha)$ *if and only if the analytic function given by* $\psi(t) = \sum a_n t^n$ *has a singular point at* α.

The proof is similar to that of Theorem 1. We again find an integral operator giving $\phi(z)$ in terms of $\psi(t)$ and conversely. Indeed

$$\begin{aligned}\phi(z) &= \int_0^\infty e^{-zx} \frac{1}{2\pi i \sqrt{2\pi}} \int_\Gamma e^{-(x/2)(t+1)/(t-1)} \psi(t) \frac{dt}{t}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_\Gamma \frac{\psi(t)}{z - \dfrac{1}{2}\dfrac{1+t}{1-t}} \frac{dt}{t} \end{aligned} \tag{8}$$

is the operator in one direction. Here we have used (7) with t replaced by $1/t$ and a contour Γ around the unit circle but inside the disk of convergence of $\sum a_n t^n$. The uniform convergence to justify the manipulation again is satisfied for $\mathcal{R}e\, z$ large enough.

To go in the other direction we first calculate

$$\begin{aligned}\sum a_n h_{2n}^2(0)\, t^n &= \int_0^\infty \sum_n h_{2n}(0)\, h_{2n}(x^{1/2})\, t^n \sum_k a_k h_{2k}(0)\, h_{2k}(x^{1/2})\, x^{-1/2}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(x/2)(t+1)/(t-1)} f(x)\, x^{-1/2}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(x/2)(t+1)/(t-1)} \frac{1}{2\pi i} \int_\Gamma e^{zx} \phi(z)\, dz (x^{-1/2})\, dx \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2\pi i} \int_\Gamma \frac{\phi(z)}{\sqrt{-z - \dfrac{1}{2}\dfrac{t+1}{t-1}}}\, dz, \end{aligned} \tag{9}$$

where Γ encloses the conjugate diagram of $f(x)$. (We have used the fact that the Laplace transform of $x^{-1/2}$ is $s^{-1/2}$). The Hadamard argument (of Theorem 1) and (8) tells us that if ψ has singular point only at α, ϕ can have one only at $\frac{1}{2}(1+\alpha)/(1-\alpha)$. The same argument and (9) tells us that if ϕ has a singular point only at $\frac{1}{2}(1+\alpha)/(1-\alpha)$, the analytic extension of $\sum a_n h_{2n}(0)\, t^n$ can have one only at α. We then use the original Hadamard "multiplication of singularities" argument applied to power series and the fact that $\sum h_{2n}^{-2}(0)\, t^n$ has a singular point only at $t = 1$ on the unit circle to go from $\sum a_n h_{2n}^2(0)\, t^n$ to $\sum a_n t^n$.

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